Distances in Sierpiński Triangle Graphs

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joint work with Andreas M. Hinz

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HÁSKÓLI ÍSLANDS
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Motivation

- **Sierpiński triangle**
  introduced by Wacław Sierpiński in 1915.

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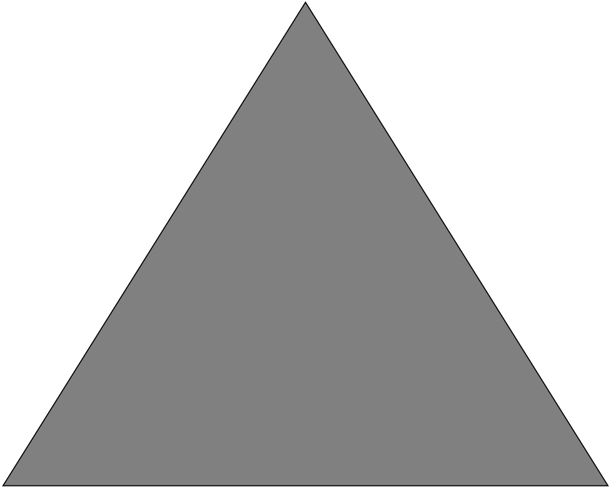

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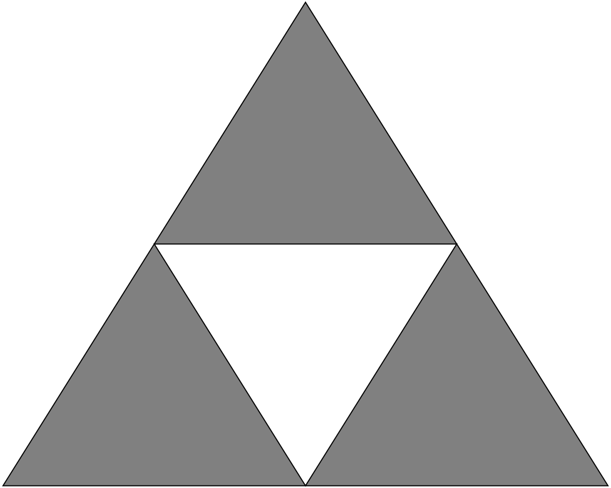

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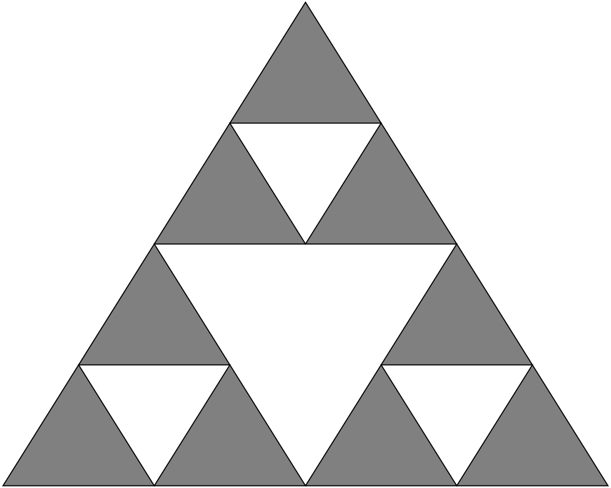

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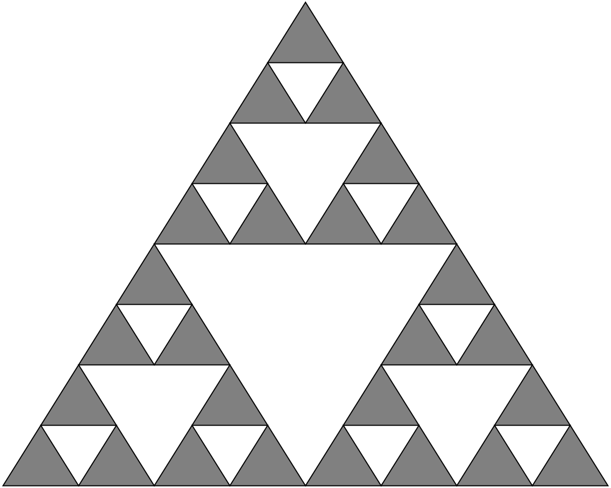

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- **Sierpiński graphs**
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  connected to the Tower of Hanoi puzzle – state graphs for the Switching Tower of Hanoi puzzle.

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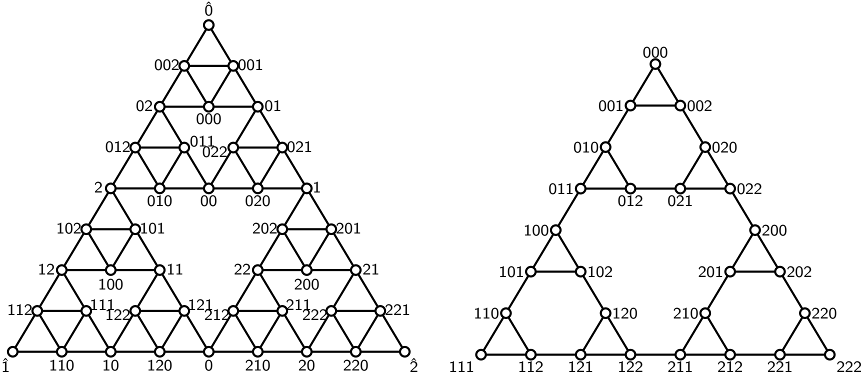

Graphs ST_3^3 (left) and S_3^3 (right).

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- **Applications outside mathemtics**
  - Physics – spectral theory (Laplace operator), spanning trees (Kirchhof's Theorem),
  - Psychology – "state graphs" of the Tower of Hanoi puzzle.

Notations

- $[n] := \{1, \ldots, n\}$,
- $[n]_0 := \{0, \ldots, n-1\}$,
- $T := [3]_0 = \{0, 1, 2\}$,
- $\widehat{T} := \{\hat{0}, \hat{1}, \hat{2}\}$,
- $P := [p]_0 = \{0, \ldots, p-1\}$,
- $\widehat{P} := \{\hat{k} \mid k \in P\}$.

Definition (Idle peg labeling)

Let $n \in \mathbb{N}$.

Sierpiński triangle graphs ST_3^n ...

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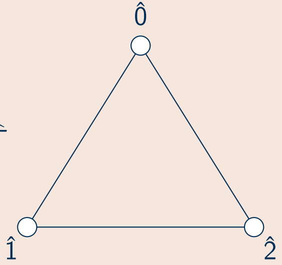
$ST_3^0 \cong K_3$

$V(ST_3^0) = \widehat{T}$



- vertices $\hat{0}$, $\hat{1}$, and $\hat{2}$ are **primitive vertices**

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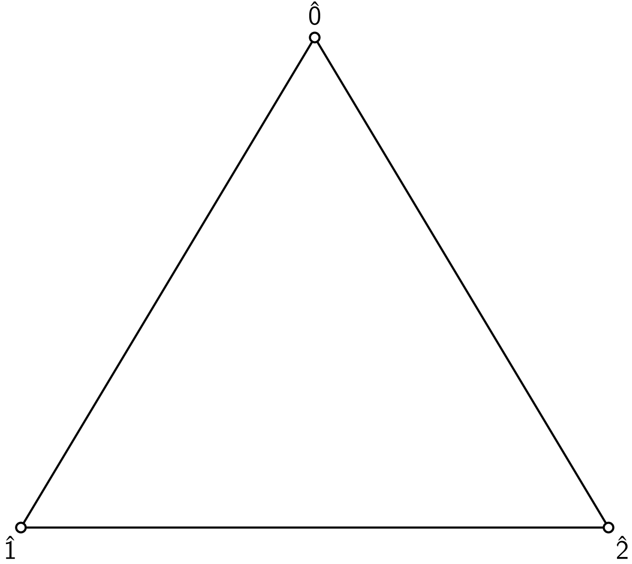
$$V(ST_3^n) = \widehat{T} \cup \{s \in T^\nu \mid \nu \in [n]\} ,$$

$$\begin{aligned} E(ST_3^n) = {} & \Big\{\{\hat{k}, k^{n-1}j\} \mid k \in T,\ j \in T \setminus \{k\}\Big\} \cup \Big\{\{\underline{s}k, \underline{s}j\} \mid \underline{s} \in T^{n-1},\ \{j,k\} \in \tbinom{T}{2}\Big\} \\ & \cup \Big\{\{\underline{s}(3-i-j)i^{n-1-\nu}k, \underline{s}j\} \mid \underline{s} \in T^{\nu-1},\ \nu \in [n],\ i \in T,\ j,k \in T \setminus \{i\}\Big\} \end{aligned}$$

Example – Idle peg labeling



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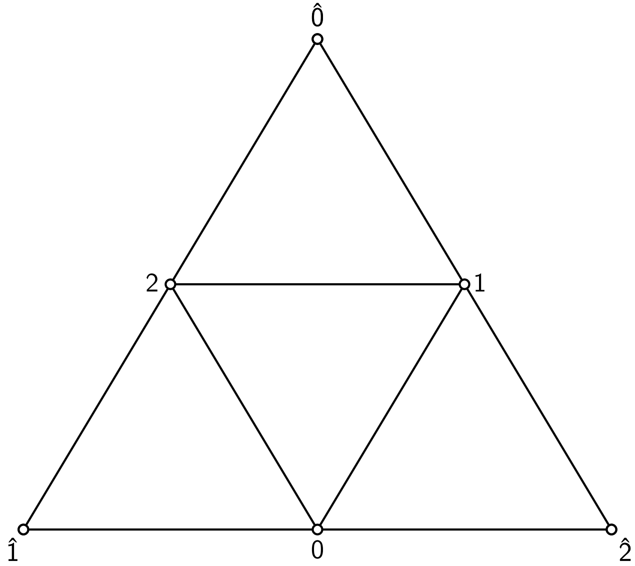

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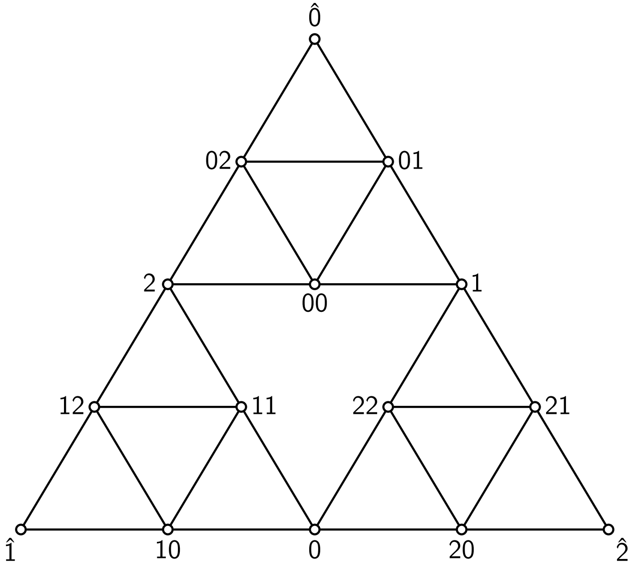

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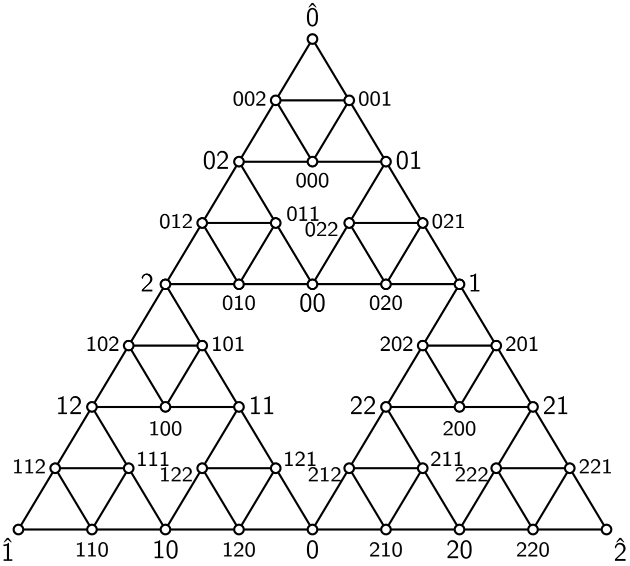

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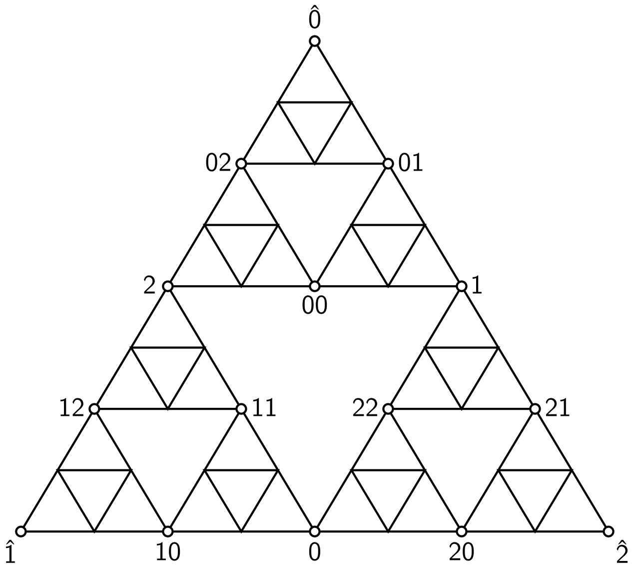

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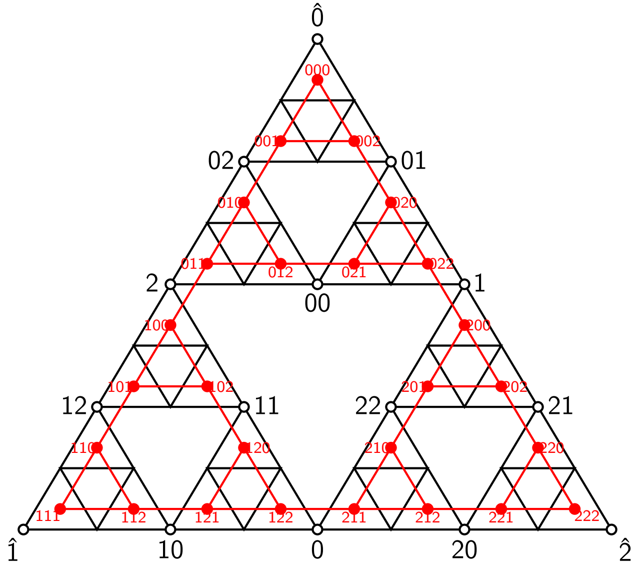

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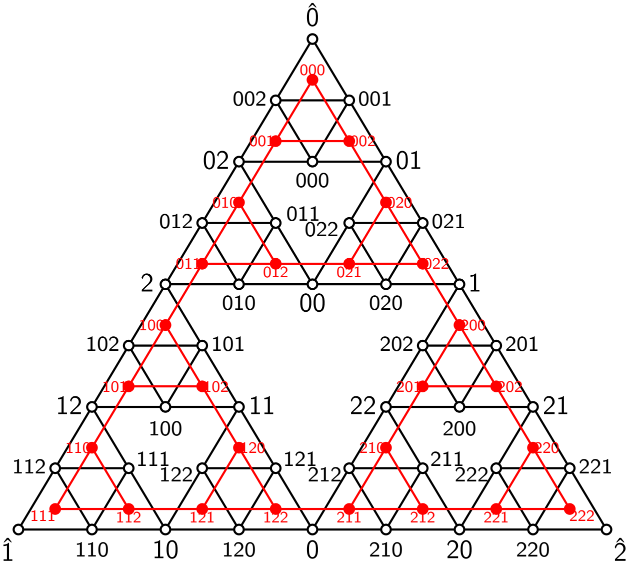

Contraction labeling

Let $n \in \mathbb{N}$.

Contraction labeling of Sierpiński triangle graphs ST_3^n

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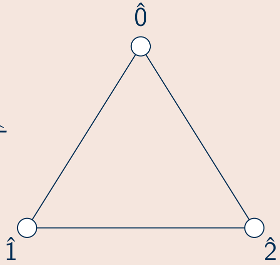
$V(ST_3^0) = \widehat{T}$



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$$E(ST_3^n) = \left\{ \{\hat{k}, k^{n-1}\{j,k\}\} \mid k \in T,\ j \in T \setminus \{k\} \right\} \cup$$
$$\left\{ \{\underline{s}\{i,j\}, \underline{s}\{i,k\}\} \mid \underline{s} \in T^{n-1},\ i \in T,\ \{j,k\} \in \binom{T\setminus\{i\}}{2} \right\} \cup$$
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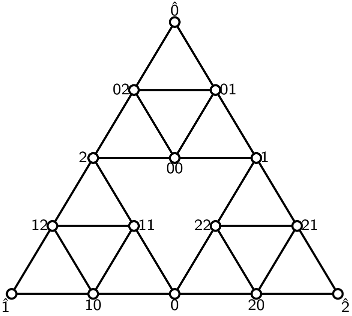
Basic properties

- $|ST_3^n| = \frac{3}{2}(3^n + 1)$
- $\|ST_3^n\| = 3^{n+1}$
- graphs ST_3^n are connected



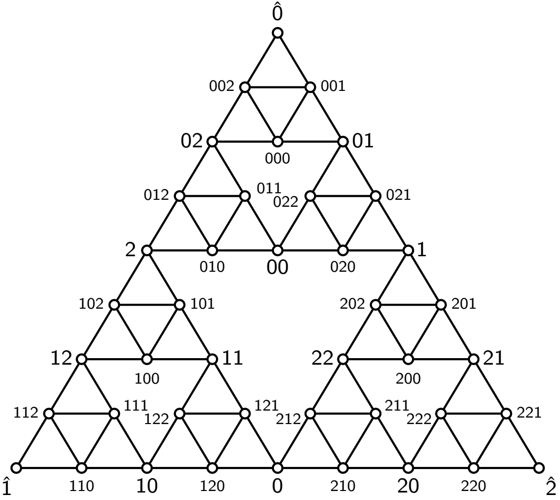
Distance to a primitive vertex

Lemma.

If $n \in \mathbb{N}$ and $\nu \in [n]_0$, then for any $s, t \in V(ST_3^\nu)$

$$\mathrm{d}_n(s,t) = 2^{n-\nu}\mathrm{d}_\nu(s,t)\,.$$

Distance to a primitive vertex



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Proposition.

If $\nu \in \mathbb{N}$ and $s \in T^\nu$, then $\mathrm{d}_0(\hat{k},\hat{\ell}) = (k \neq \ell)$, and

$$\mathrm{d}_\nu(s,\hat{\ell}) = 1 + (s_1 = \ell) + \sum_{d=2}^{\nu}(s_d \neq \ell)\cdot 2^{d-1}\,.^a$$

There are $1 + (s_1 = \ell)$ shortest paths between s and $\hat{\ell}$.

[a] Here (X) is Iverson convention, which is 1 if X is true and 0 if X is false.

Distances – special case

Let $\{i,j,k\} = T$, $n \in \mathbb{N}$ and $s \in T^n$.

$$\mathrm{d}_{n+1}(is,j) = \mathrm{d}_n(s,\hat{k})$$
$$\mathrm{d}_{n+1}(is,i) = \min\{\mathrm{d}_n(s,\hat{k}) \mid k \in T \setminus \{i\}\} + 2^n$$

If $s = i^\kappa s_{n-\kappa}\overline{s}$, $\kappa \in [n-1]_0$, then $\mathrm{d}_{n+1}(is,i) = \mathrm{d}_n(s,\widehat{s_{n-\kappa}}) + 2^n$ and the shortest path goes through vertex $3-i-s_{n-\kappa}$.

- two shortest paths between is and i iff $is = i^{\nu+1}$, $\nu \in [n]$
- two shortest paths between is and j iff $is = ik^\nu$, $\nu \in [n]$

Distances – general formula

Theorem.

If $n \in \mathbb{N}$ and $\nu \in [n]_0$, then for any $s \in V(ST_3^n)$, $t \in V(ST_3^\nu)$, and $\{i, j, k\} = T$,

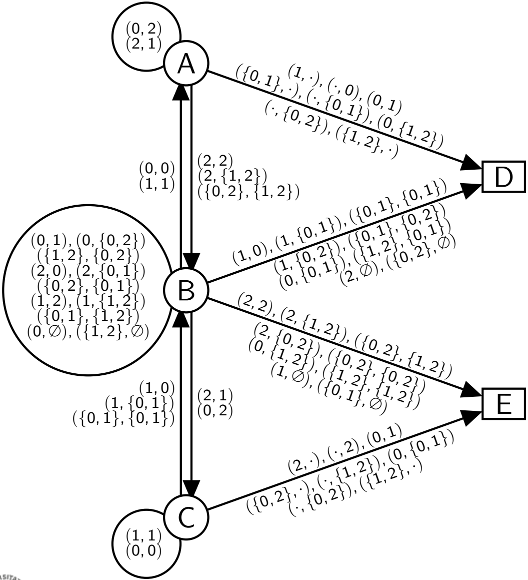
$$d_{n+1}(is, jt) = \min\{d_n(s, \hat{j}) + 2^{n-\nu} d_\nu(t, \hat{i})\,;\, d_n(s, \hat{k}) + 2^n + 2^{n-\nu} d_\nu(t, \hat{k})\}\,.$$

Problem of two shortest paths: shortest path either goes directly from i-subgraph to j-subgraph, or it goes through k-subgraph. It can also happen that there are two shortest paths.

Comparison
with metric properties of Sierpiński graphs

S_3^n	ST_3^n
$d(\underline{s}\overline{s}, \underline{s}\overline{t}) = d(\overline{s}, \overline{t})$	$d_n(s,t) = 2^{n-\nu} d_\nu(s,t)$
$d(s, j^n) = \sum_{d=1}^{n} (s_d \neq j) 2^{d-1}$	$d_\nu(s, \hat{\ell}) = 1 + (s_1 = \ell) + \sum_{d=2}^{\nu} (s_d \neq \ell) 2^{d-1}$
$d(is, jt) = \min\{d_{dir}(is, jt), d_{indir}(is, jt)\}$	
$\mathrm{diam}(S_3^n) = 2^n - 1$	$\mathrm{diam}(ST_3^n) = 2^n$
$\sum_{i \in T} d(s, i^n) = 2^{n+1} - 2$	$\sum_{i \in T} d(s, i^n) = 2^{n+1}$

Automaton



Example

$d_4(002\{0,2\}, 112\{1,2\}) = 16$
(direct)

$d_4(020\{1,2\}, 12\{0,2\}) = 13$
(two shortest paths)

$d_4(022\{0,1\}, 12\{0,2\}) = 12$
(indirect)

Sierpiński triangle graphs ST_p^n

Jakovac, **A 2-parametric generalization of Sierpiński gasket graphs**, <u>Ars. Combin.</u> 116 (2014) 395–405.

Sierpiński triangle graphs ST_p^n $(n \in \mathbb{N})$ ...

... are the graphs defined by:

$$V(ST_p^n) = \widehat{P} \cup \left\{ \underline{s}\{i,j\} \mid \underline{s} \in P^{\nu-1},\ \nu \in [n],\ \{i,j\} \in \binom{P}{2} \right\},$$

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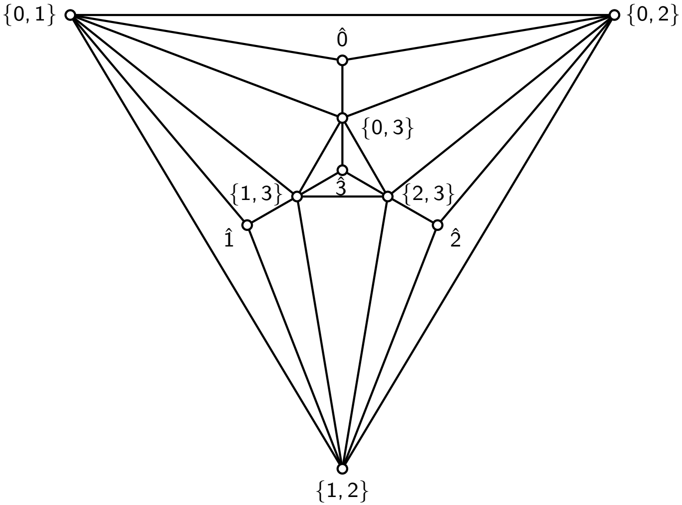
As before, $ST_p^0 \cong K_p$ and $V(ST_p^0) = \widehat{P}$.

Example ST_4^1



Distances in ST_p^n

$$\mathrm{d}_\nu(\underline{s}\{i,j\},\hat{\ell}) = 1 + (i \neq \ell)(j \neq \ell) + \sum_{d=1}^{\nu-1} (\underline{s}_d \neq \ell) \cdot 2^d \in [2^\nu]$$

- there are $1 + (p-2)(i \neq \ell)(j \neq \ell)$ $\underline{s}\{i,j\},\hat{\ell}$-shortest paths
- $\mathrm{diam}(ST_p^n) = 2^n$
- $\forall\, s \in V(ST_p^n) : \sum_{\ell=0}^{p-1} \mathrm{d}_n(s,\hat{\ell}) = (p-1) \cdot 2^n$

$$\mathrm{d}_{n+1}(is,jt) = \min\{\mathrm{d}_n(s,\hat{j}) + 2^{n-\nu}\mathrm{d}_\nu(t,\hat{\imath})\,; \\ \mathrm{d}_n(s,\hat{k}) + 2^{n-\nu}\mathrm{d}_\nu(t,\hat{k}) + 2^n \mid k \in P \setminus \{i,j\}\}\,.$$

Open Problems

- explicit formula for average distance
- other metric properties which are known for Sierpiński graphs S_p^n
- we are currently working on the decision automaton for $p > 3$

THANK YOU!